THE MATHEMATICAL GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.
WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc
AND
PROF. E. T. WHITTAKER, M.A., F.R.S.

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THE STORY OF MERCATOR'S MAP.

A CHAPTER IN THE HISTORY OF MATHEMATICS.

BY PROF. H. S. CARSLAW, M.A., Sc.D.

1. In 1569 Gerhardt Mercator (1512-1594) constructed the chart now called Mercator's Map. On it the meridians and parallels are sets of parallel straight lines at right angles, a course on the Earth's surface always cutting the meridians at the same angle is a straight line cutting the meridians on the map at the same angle, and the shape of any very small area on the map is the same as the shape of the corresponding small area upon the Earth. This chart can be made by gradually increasing the distances between the degrees of latitude in advancing from the equator towards the poles. But though Mercator did, in fact, construct such a map, it is not known on what principle he proceeded, and it is conjectured that he obtained his division of the meridians by observing on a globe on which rhumb-lines * had been drawn the intersection of these lines with the different meridians.

2. Probably Edward Wright (1558?-1615) is best known as the translator into English of *Napier's Mirifici Logarithmorum Canonis Descriptio* (1614), but he should also be kept in remembrance for his discovery of the true way of dividing the meridian in Mercator's Map. "Of this discovery Mr. Wright sent an account from Caius College in Cambridge (of which College he was then a Fellow) to his friend ... Mr. Blundeville, containing a short table for that purpose, with a specimen of a chart so divided, together with the manner of dividing it: All which Blundeville published, in 1594, amongst his *Exercises*, in that part of them which treats of Navigation; where he has well delivered what had been before written on that Art." †

Wright was rather proud of his discovery, and in the preface to his work—*Certaine Errors in Navigation ... Detected and Corrected* (1599)—he defends himself at length from probable critics‡—the seafaring men who "are ashamed peradventure to receive (as it were) either correction from the schooles, or

* *Rhumb* is an old word for a compass point, and a rhumb-line is a course always directed to the same point of the compass.

† Cf. Maseres, *Scriptores Logarithmici*, vol. 4, p. 314 (1791-1807) (from Wilson's *Dissertation on the Rise and Progress of Navigation*).

‡ Cf. *Certaine Errors in Navigation*, p. 9.

direction from the land; and therefore stick not to condemne Universities," and from others who may say that he is "doing no more than hath bin done alreadie by Gerhardus Mercator in his Universalle Mappe of the World many yeers since." To these he must answer that "indeed by occasion of that Mappe of Mercator he first thought of correcting so many, and grosse errors, and absurdities . . . in the common Sea Chart, by increasing the distances of the Parallels from the Aequinoctial towards the Poles, in such sort, that at every point of Latitude in the Chart, a small part of the Meridian might have the same proportion almost to the like part of the Parallel, that it hath in the Globe. But the way how this should be done, he learned neither of Mercator nor of any man else." And in that point he wishes he had been as wise as Mercator in keeping it more charily to himself. The fact being that at the instant request of Hondius, a member of a Dutch family of cartographers then working in London, he had shown his manuscript book to him, Hondius also assuring him, upon his faith and credit, that he would not publish it, or any part thereof, without Wright's knowledge and consent. But the said Hondius published in Amsterdam maps of the World, of Europe, of Asia, Africa and America, "all which had yet been unhatched, had he not learned the right way to lay the ground work of them out of this booke. . . . But let him goe as he is."

3. It would take too long to quote Wright's description of the way he conceives the surface of the globe to be deformed into the plane of the chart. But he concludes that "the parts of the Meridian at every point of latitude must needs increase with the same proportion wherewith the Secants of the ark, conteined between these points of latitude and the Aequinoctial do increase." In modern notation, the element of the meridian at latitude θ being $r\,d\theta$ and the element of the parallel being $r\cos\theta\,d\phi$, as the element of the parallel is increased on the chart in the ratio $\sec\theta : 1$, the element of the meridian is to be increased in the same ratio.

In making his table of meridional parts Wright in effect finds by summation the definite integral $\int_0^\theta \sec\theta\,d\theta$.* "The secans of one minute is 10,000,000,† which also sheweth the section of one minute of the Meridian from the Aequinoctial in the nauticall planisphaere. Whereunto adde the secans of 2. minutes, that is, 10,000,002, the sum is 20,000,002, which sheweth the section of the second minute of the Meridian from the Aequinoctial, in the planisphaere; to this sum adde the secans of 3. minutes, which is 10,000,004, the sum will be 30,000,006, which sheweth the section of the third minute of the Meridian from the Aequinoctial; and so forth in all the rest: saving that in this table we have of set purpose omitted in every secans the three first cyphers next the right hand; not only for the easier, but also for the truer making of the table, because that indeed, at every point of latitude, a minute of the Meridian in this nautical planisphaere, hath somewhat lesse proportion to a minute of the Parallel adjoyning towards the Aequinoctial, than the secans of that Parallel's latitude hath to the whole sine."

So here we have the Table of Meridional Parts, as, for example, to be found in Chambers' Tables, the only difference between the two being that in Wright's Table (I quote from the edition of 1657) ‡ the arc of a minute of the equator is represented by 10,000, whereas in Chambers' Tables it is given by 1·0.

* Cf. Cajori, "On an integration antedating the Integral Calculus," *Bibliotheca Mathematica* (3 F.), Bd. 14, p. 312, 1914. And a paper by the same author in the *Napier Tercentenary Memorial Volume*, p. 95, 1915.

† In those days the sine, cosine, etc., were lines in a circle of given radius. Sometimes this radius was taken to be 10,000,000, as in Wright's work. If greater accuracy was needed, this was replaced by 10,000,000,000. The sine of 90° is equal to the radius, and is called the *whole sine* or *sinus totus*.

‡ In the first edition of Wright's work (1599) the meridian was divided only to every ten minutes. In the later editions the division was to every minute.

4. About the year 1645 it seems to have been observed that the length of the meridian line for θ°, as given in the Table of Meridional Parts, bears to the length for 1° the same ratio as $\log\tan(45^\circ + \frac{1}{2}\theta^\circ)$ to $\log\tan 45^\circ\, 30'$. But no proof of this equality was known. However, by 1666 Nicholas Mercator (1620?-1687) had made a considerable advance towards the solution of the problem. For in a communication* to the Royal Society of London—founded in 1660—referring to the importance of an exact determination of the divisions of the meridian, he says: "Seeing all these things do depend on the solution of the Question: *Whether the Artificial Tangent*† *Line be the true Meridian Line*; it is therefore that I undertake, by God's Assistance, to resolve the said Question. And, to let the World know the Readiness and Confidence I have to make good this Undertaking, I am willing to lay a Wager against any one or more Persons that have a Mind to engage, for so much as another Invention of mine (which is of less Subtilty, but of a far greater Benefit to the Publick) may be worth to the Inventor."

This Nicholas Mercator was a native of Holstein, who spent most of his life in London, where he was well known as a mathematician. In 1668 he published *Logarithmo-technia: sive methodus construendi logarithmos nova, accurata, et facilis; scripto antehac communicata, anno sc.* 1667, *nonis Augusti. Cui nunc accedit vera quadratura hyperbolae, et inventio summae logarithmorum.*

In Prop. XVII. he finds the area included between a rectangular hyperbola, an asymptote, and the ordinates to this asymptote with abscissae 1 and $(1+x)$; and though he does not explicitly give the logarithmic series

$$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \dots,$$

it is clear from Prop. XIX.—*invenire summam logarithmorum*—(*i.e.*, in modern notation, to find the integral $\int_0^x \log(1+x)\,dx$)—that this expansion was known to him.

It is true that the area above referred to had been discovered earlier. For Gregory St. Vincent ‡ had shown that, if the abscissae increase in geometrical progression, the areas increase in arithmetical progression. And this is the fundamental property of logarithms as understood in Napier's time and long after. So that it was known that, starting from the abscissa unity, the hyperbolic area was equal to $\log(1+x)$ in some system of logarithms.

But the *Logarithmo-technia* will always be notable, not only because of its connection with the logarithmic series, but also because in it we first find infinite series appearing in analysis. Mercator is dealing with the hyperbola whose semi-axes are each $\sqrt{2}$. The asymptotes are taken as axes. If y is the ordinate of the point whose abscissa is $(1+x)$, we have $y(1+x)=1$.

Thus $$y = \frac{1}{1+x}.$$

Dividing out by $(1+x)$, he obtains the infinite series

$$y = 1 - x + x^2 - x^3 + \dots.$$

Now the quadrature of the curves of the type $y = x^n$, where n is a positive integer, was already known, the area being obtained by a summation equivalent to integration. Applying this method to the different terms of the series for $(1+x)^{-1}$, the area from 1 to $(1+x)$ is given in the form

$$x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \dots.$$

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* No. 13, *Phil. Trans.* 1666. The quotations are taken from the version in the *Abridged Transactions* issued in 1722 by Lowther.

† Napier in the *Constructio* (1616) used the term *artificial number* instead of *logarithm*. And this practice was followed by some writers. In the *Descriptio*, written later than the *Constructio* but published in 1614, he used the term *logarithm* throughout.

‡ For Gregory St. Vincent's work on the Conic Sections, see *Abh. zur Geschichte der math. Wiss.* Heft xx. 2 Stück, 1907. This property of the hyperbola is given in Prop. CIX. See also CXXV.-CXXX.

Thus it was known that

$$\log(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\ldots.$$

He says nothing about the need for x being less than unity; indeed, in his diagram it is greater than unity.

It is quite possible that Mercator was led to this investigation by his interest in the division of the meridian. He published nothing further on the subject, but we shall see that the ideas contained in the *Logarithmo-technia* attracted the attention of other mathematicians, and that his introduction of infinite series was responsible for the solution of the problem of the division of the meridian line and for other much more important discoveries.

5. For we have now come to the time of Newton (1642-1727), and the invention of the Infinitesimal Calculus. Newton entered Trinity College, Cambridge, in 1660. Isaac Barrow (1630-1677), on his election to the newly created Lucasian Chair of Mathematics in 1663, returned to Trinity, of which he had been elected a Fellow in 1649. He resigned his chair in 1669 in Newton's favour, and for the remainder of his life devoted himself to the study of Divinity. By mathematicians he is probably best known as the teacher of Newton, but his own mathematical writings are of great interest, and his *Lectiones Geometricae* (1670) is a most important work. Newton shared in the revision of the book for the press, "adding some things from his own work, which you will see annexed with praise here and there." It is supposed that the sections in Lecture X., added, as he says, "on the advice of a friend," were, like other passages in the text where a similar expression is used, inserted on the advice of Newton.

There is, of course, no suggestion of any rivalry between the two. We shall see in the next paragraph that Barrow first brought Newton's work to the attention of mathematicians; and though Barrow's *Lectures* certainly do contain, almost always in geometrical form, many of the results of both the Differential and Integral Calculus, a recent attempt * to claim for Barrow the credit of the invention of the Calculus, rather than Newton and Leibniz, seems to me, at least, to rest on a very slender foundation.

Barrow, however, knew Mercator's *Logarithmo-technia*, as is clear from the following letter to Collins (1625-1683), whose name is familiar to all students of the Newton-Leibniz controversy.

On July 20, 1669, he writes from Cambridge: † "A friend of mine here, that hath an excellent Genius to these Things, brought me the other Day some Papers, wherein he hath set down some Methods of calculating the Dimensions of Magnitudes, like that of Mr. Mercator for the Hyperbola, but very general, as also of Resolving Equations, which I suppose will please you, and I shall send them by the next."

Then on July 31 he continues: "I send you the Papers of my Friend I promis'd, which I presume will give you much Satisfaction: I pray, having perused them so much as you think good, remand them to me, according to his desire, when I ask'd him the Liberty to impart them to you; I pray give me notice of your receiving them, with your soonest Convenience, that I may be satisfied of their Reception; because I am afraid of them, venturing them by the Post, that I may not longer delay to correspond with your desire."

And a third letter, written three weeks later, discloses the name of the author:

"I am glad my Friend's Paper gives you so much Satisfaction: his name is Mr. Newton, a Fellow of our College, and very young, (being but the second Year Master of Arts) but of an extraordinary Genius and Proficiency in these Things: you may impart the Papers if you please to my Lord Brouncker." ‡

* Cf. Child's translation of the *Lectiones Geometricae*, Chicago, 1916.

† Collins, *Commercium Epistolicum*, 1712.

‡ Brouncker was one of the founders and the first President of the Royal Society.

6. It will be remembered that Newton was never in a hurry to publish his writings, and though the *Principia* did not appear till 1687, there can be little doubt that he was in possession of his new methods from about 1666. The papers to which Barrow refers in these letters were not published till 1711. However, Collins made a copy of the work, and its contents soon became known to mathematicians through him. It was entitled *De Analysi per aequationes numero terminorum infinitas*. The number and importance of the series it contains are very striking. Besides the binomial series, for the case in which n is not a positive integer, which he refers to as the method of the "extraction of roots," it contains the series for $\sin^{-1}x$, $\sin x$, $\cos x$, $\log(1+x)$, and what we now call the exponential series. For the quadrature of curves he depends upon a theorem involving the method of fluxions. In our notation we are familiar with it in the form

$$\frac{dA}{dx}=y,$$

A being the area from a fixed point up to the point (x, y). He is ashamed to say to what number of figures he evaluated the logarithms of various numbers from the series for $\log(1+x)$: *e.g.* from those for 0·8, 0·9 and 1·2 he found $\log 2$, since $(1{\cdot}2)^2=0{\cdot}8\times 0{\cdot}9\times 2$. Similarly from the equation $10\times 0{\cdot}8=2^3$ he found $\log 10$.

But when he heard from Barrow of Mercator's *Logarithmo-technia* with its method of division, he supposed its author would also have discovered the method of "extraction of roots," or, if he had not done so, that it would be soon found out by some other person "before he would be of an age for publishing." So he did not trouble himself further in the matter.

7. James Gregory (1638-1675), the Aberdonian, who was successively Professor of Mathematics at St. Andrews and Edinburgh, published in 1668 a set of mathematical tracts under the title *Exercitationes Geometricae.* One of them contains a geometrical demonstration of Mercator's expression for the area of a part of the hyperbola, and another deals with the "analogy"* between the divisions of the meridian and the "artificial tangents."

But early in 1670 Collins told Gregory Newton's series for the area of a part of a circle (in our notation $\int_0^x \sqrt{(a^2-x^2)}\,dx$). Gregory spent much time on the question, and it was not till December that he was able to inform Collins that he had arrived at the same result. In the same letter he gave Collins the series for $\sin^{-1}x$, only to learn from his correspondent that this expansion, as well as those for $\sin x$ and $\cos x$, was given in the *De analysi*. Gregory in the following February tells Collins he thinks he knows Newton's method, and forwards some further results of his own, namely the series for $\tan^{-1}x$ (now called Gregory's Series), and those for $\tan x$, $\sec x$, $\log\tan x$ and $\log\sec x$.

8. In following the story of Mercator's problem we now come to the work of John Wallis (1616-1703) of Emmanuel College, Cambridge, later Savilian Professor of Geometry at Oxford. In 1668 Wallis communicated to the Royal Society an account † of the *Logarithmo-technia*, with another infinite series for the quadrature of the hyperbola, a method of finding the sums of logarithms, and an explanation of the said tract by Mercator himself. All this in a letter to Lord Brouncker, the President, beginning thus: "*Incidebam heri, Illustrissime Domine, in D. Mercatoris Logarithmo-techniam, nupèr editam; quae ita mihi placuit, ut non prius dimiserim quàm perlegissem totam.*"

But Wallis is rather vague about Mercator's summation of logarithms, and his reply to Brouncker's request for a clearer exposition is not very satisfying.

* In those days "analogy" meant "proportion," as for instance in Napier's *Analogies*.

† No. 38, *Phil. Trans.* 1668. In the *Abridged Transactions* only the title of this paper appears. The paper itself will be found in Maseres, *loc. cit.* vol. i.

Seventeen years later, in the *Phil. Trans.*,* we find him returning to the problem in a letter to a certain Mr. Richard Norris "concerning the Collection of Secants, and the true Division of the Meridians in the Sea Chart." His paper, which is somewhat difficult to follow, is an attempt to obtain the summation involved in the integral $\int \sec^2\theta \, d(\sin\theta)$.

Expanding $\sec^2\theta$ as $1+s^2+s^4+\ldots$, where $s=\sin\theta$, and integrating with regard to s, or in his language "according to the arithmetic of infinites," we have, in our notation,

$$\int_0^\theta \sec\theta \, d\theta = \tfrac{1}{2}\log\frac{1+\sin\theta}{1-\sin\theta}.$$

As a matter of fact this result had already been given by Barrow in his *Lectiones Geometricae*, Prop. XII., App. I. § 5, but from the geometrical form of Barrow's work it is not altogether surprising that Wallis was unaware that he was "putting his sickle into another man's harvest."

9. Before closing this account of the successive attempts to establish the truth of Mercator's Theorem, we must mention a paper † by Edmund Halley (1656-1742), who succeeded Wallis in the Savilian chair in 1703. His name is familiar from its association with the famous comet, whose return in 1758, after its appearances in 1531, 1607 and 1682, he accurately predicted; and mathematicians will always remember his share in the publication of Newton's *Principia*.

The paper to which we refer is entitled: "*An easy Demonstration of the Analogy of the Logarithmick Tangents to the Meridian Line, or Sum of the Secants; with various Methods for computing the same to the utmost Exactness.*" From the introduction the following passage may be quoted: "The first that demonstrated the said analogy was the excellent Mr. James Gregory, in his *Exercitationes Geometricae*, published *anno* 1668; which he did not without a long train of consequences, and complication of proportions, whereby the evidence of the demonstration is in a great measure lost, and the reader wearied before he attain it. Nor with less work and apparatus hath the celebrated Dr. Barrow in his Geometrical Lectures (Lect. XII. App. I.), proved, that the sum of all the secants of any arch is analogous to the logarithm of radius + sine to rad. − sine; or, which is all one, that the meridional parts answering to any degree of latitude, are as the logarithms of the rationes of the versed sines of the distances from both the poles. Since which, the incomparable Dr. Wallis (on occasion of a paralogism committed by one Mr. Norris in this matter), has more fully and clearly handled this argument, as may be seen in number 176 of the Transactions. But neither Dr. Wallis nor Dr. Barrow, in their said Treatises, have anywhere touched upon the aforesaid relation of the meridian line to the logarithmick tangent. . . . Wherefore having attained, as I conceive, a very facile and natural demonstration of the said analogy. . . . I hope I may be intitled to a share in the improvements of this useful part of geometry."

The demonstration to which Halley refers is a geometrical one. He showed that if the sphere is projected stereographically from a pole on the plane of the equator, the rhumb-lines become equiangular spirals. The properties of this curve were already known, and by means of these he proves what we now express by the equation

$$\int_0^\theta \sec\theta \, d\theta = \log\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right).$$

* No. 176, *Phil. Trans.* 1685. Cf. *Abridged Transactions*, vol. i. p. 572.

† No. 219, *Phil. Trans.* 1696. Cf. *Abridged Transactions*, vol. i. p. 577

Towards the close of his paper he gave this other demonstration, interesting from its use of Newton's series for the trigonometrical functions and the different notation of these days: "To conclude, I shall only add that unity being radius, the cosine of the arch A, according to the same rules of Mr. Newton, will be

$$1-\tfrac{1}{2}A^2+\tfrac{1}{24}A^4-\tfrac{1}{720}A^6+\tfrac{1}{40320}A^8-\tfrac{1}{3628800}A^{10}, \text{ etc.};$$

from which, and the former series exhibiting the sine of the arch, by division, it is easy to conclude, that the natural tangent of the arch A is

$$A+\tfrac{1}{3}A^3+\tfrac{2}{15}A^5+\tfrac{17}{315}A^7+\tfrac{62}{2835}A^9, \text{ etc.};$$

and the natural secant to the same arch

$$1+\tfrac{1}{2}A^2+\tfrac{5}{24}A^4+\tfrac{61}{720}A^6+\tfrac{277}{8064}A^8, \text{ etc.};$$

and from the arithmetick of infinites, the number of these secants being the arch A, it follows, that the sum total of all the infinite secants on that arch is

$$A+\tfrac{1}{6}A^3+\tfrac{1}{24}A^5+\tfrac{61}{5040}A^7+\tfrac{277}{72576}A^9, \text{ etc.};$$

the which, by what foregoes, is the logarithm tangent of Napier's form, for the arch of 45 gr. $+\tfrac{1}{2}A$, as before."

10. The story of Mercator's projection illustrates the importance of a suitable notation to the development of mathematics. If English mathematicians had adopted the differential notation, men of the standing of Mercator, Wallis, Gregory and Halley would not have spent so much time on a problem which after all reduced simply to the integration of the secant. Leibniz recognised that he had found in his d and symbols of immense value. And the development of analysis in England was delayed for many years by the refusal of the followers of Newton to avail themselves of the instrument offered to them by his great contemporary.

H. S. Carslaw.

GLEANINGS FAR AND NEAR.

201. Erasmus, referring to Cuthbert Tunstall, in a letter to Peter Gillis. Brussels, Jan. 1517: "Our age does not possess a more learned, a better, or a kinder man."

202. Poor Jones! I am sure there is not a man we could have less spared, or one who is more lamented by our College or by the University in general.—Sedgwick's *Life and Letters*.

[Jones was a tutor of Trinity College, Cambridge.]

203. I am, potentissima Domina, a schoolmaster—that is to say, a pedagogue—one not a little versed in the disciplinating of the juvenile fry, wherein, to my laud I say it, I use such geometrical proportion as neither wanteth mansuetude nor correction, for so it is described

"Parcare subjectos et debellire superbos."

—"Rhombus" addresses the Queen in Sidney's masque: *The Lady of the May*, 1573.

204. Our countryman Norwood set about determining the circumference of the earth, with an accuracy as much superior to that of the Greek geometer as it was inferior to our method of to-day. Having determined the latitudes of London and York by observation, he travelled from York to London, or vice versâ, measuring along the high road with a chain and taking the bearings with a compass. He was well satisfied with the accuracy of his work. "When I measured not I paced, and I believe the experiment has come within a scantling of the truth."—[Source lost].

SOME DEDUCTIONS FROM THE FORM OF THE EQUATION OF THE CONIC SECTION.

By L. B. Benny, M.A.

In teaching Analytical Geometry very little attention is often given to the interpretation of the form of an equation, although this may often convey a good deal of information; by grouping the terms of an equation in different ways, a variety of useful and interesting results may be obtained with very little labour.

Some suggestions as to the possibilities of this method may be afforded by the cases given below. A considerable number of results are obtained, purely by simple manipulation of the general equation of the conic section in point and line co-ordinates. Except in one instance, no reference has been made to infinite elements; if we include the notions of infinite points and lines, and of the circular points, the power of the method increases very greatly. The only knowledge that has been assumed is that which is quite fundamental in point and line co-ordinates, and the meanings of certain simple typical forms.

I. Homogeneous Point Co-ordinates.

The conic is $S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$, $A_0B_0C_0$ is the triangle of reference, $A\ldots$, $F\ldots$, are, as usual, the co-factors of $a\ldots$, $f\ldots$ in the determinant $\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, and the typical forms used are $f_2 + kuv = 0$, $f_2 + ku^2 = 0$, $w^2 + kuv = 0$, and $uv + kwr = 0$, where f_2 is quadratic and u, v, w, r are linear in the co-ordinates x, y, z. Their interpretation is sufficiently obvious.

A. We have

$$aS \equiv a^2x^2 + aby^2 + acz^2 + 2afyz + 2agzx + 2ahxy$$
$$\equiv (ax + hy + gz)^2 + y^2(ab - h^2) + z^2(ac - g^2) + 2yz(af - gh)$$

or $$aS \equiv (ax + hy + gz)^2 + Cy^2 - 2Fyz + Bz^2,$$

and similarly

$$bS \equiv (hx + by + fz)^2 + Az^2 - 2Gzx + Cx^2$$

and $$cS \equiv (gx + fy + cz)^2 + Bx^2 - 2Hxy + Ay^2.$$

By comparison with the form $w^2 + kuv = 0$, we have the following conclusions:

(i) The polars of A_0, B_0, C_0 are $ax + hy + gz = 0$, $hx + by + fz = 0$, and $gx + fy + cz = 0$ respectively, or $\frac{\partial S}{\partial x} = 0$, $\frac{\partial S}{\partial y} = 0$, and $\frac{\partial S}{\partial z} = 0$.

If S is a line-pair, these polars concur in the meet of the lines, whence on eliminating $x : y : z$, the condition for a line-pair is $\Delta = 0$. For the meet of the lines, solving two of the above equations,

$$x : y : z = G : F : C = \sqrt{AC} : \sqrt{BC} : C = \sqrt{A} : \sqrt{B} : \sqrt{C},$$

since $$BC - F^2 \equiv a\Delta = 0, \quad CA - G^2 \equiv b\Delta = 0.$$

Returning to the non-degenerate case, the polars of A_0, B_0, C_0 meet the respective opposite sides of the $\triangle A_0B_0C_0$ in the points

$$\left(x=0,\ \frac{y}{g}+\frac{z}{h}=0\right),\quad \left(y=0,\ \frac{z}{h}+\frac{x}{f}=0\right),\quad \left(z=0,\ \frac{x}{f}+\frac{y}{g}=0\right).$$

These are clearly collinear on the line $\frac{x}{f}+\frac{y}{g}+\frac{z}{h}=0$.

(ii) The pairs of tangents from the vertices of the triangle of reference are:

$$\left.\begin{array}{ll} \text{From } A_0, & Cy^2-2Fyz+Bz^2=0. \\ \text{From } B_0, & Az^2-2Gzx+Cx^2=0. \\ \text{From } C_0, & Bx^2-2Hxy+Ay^2=0. \end{array}\right\}$$

If the tangents from A_0 meet B_0C_0 in P_1, P_2, these points are given by $x=0$, $Cy^2-2Fyz+Bz^2=0$; and similarly for points Q_1, Q_2, and R_1, R_2 obtained correspondingly. Hence P_1, P_2, Q_1, Q_2, R_1, R_2 lie on the conic

$$BCx^2+CAy^2+ABz^2-2AFyz-2BGzx-2CHxy=0,$$

as is evident on putting x, y, z equal to zero in turn in this equation. Also it is easily verified that this equation may be written in the form $\Delta S+S'=0$, where

$$S'\equiv F^2x^2+G^2y^2+H^2z^2-2GHyz-2HFzx-2FGxy;$$

hence the conic $P_1P_2Q_1Q_2R_1R_2$ passes through the meets of the original conic S, and a certain conic S' inscribed in the $\triangle$ of reference (see (v) below).

(iii) If S has $A_0B_0C_0$ for a self-polar triangle, the polars of A_0, B_0, C_0 are $x=0$, $y=0$, $z=0$ respectively. Hence from (i) $f=g=h=0$, and we see that a conic for which the $\triangle$ of reference is self-polar has for its equation

$$ax^2+by^2+cz^2=0.$$

(iv) If S is circumscribed to the $\triangle A_0B_0C_0$, then we have coincident pairs of tangents from A_0, B_0, C_0. Then (ii) gives

$$BC-F^2\equiv a\Delta=0,\quad CA-G^2\equiv b\Delta=0,\quad AB-H^2\equiv c\Delta=0,\quad \text{or}\quad a=b=c=0.$$

Hence for a circumscribed conic $S\equiv fyz+gzx+hxy$.

We may put this in the forms,

$$\left.\begin{aligned} S&\equiv fyz+ghx\left(\frac{y}{g}+\frac{z}{h}\right) \\ &\equiv gzx+hfy\left(\frac{z}{h}+\frac{x}{f}\right) \\ &\equiv hxy+fgz\left(\frac{x}{f}+\frac{y}{g}\right). \end{aligned}\right\}$$

Comparing with the type $uv+kwr=0$, it follows that the tangents to S at A_0, B_0, C_0 are

$$\frac{y}{g}+\frac{z}{h}=0,\quad \frac{z}{h}+\frac{x}{f}=0,\quad \frac{x}{f}+\frac{y}{g}=0 \text{ respectively.}$$

Further, the points L, M, N, in which these tangents meet the respective opposite sides of the $\triangle A_0B_0C_0$, are

$$\left(x=0,\ \frac{y}{g}+\frac{z}{h}=0\right),\quad \left(y=0,\ \frac{z}{h}+\frac{x}{f}=0\right),\quad \text{and}\quad \left(z=0,\ \frac{x}{f}+\frac{y}{g}=0\right) \text{ respectively.}$$

Hence LMN are collinear on the line $\frac{x}{f}+\frac{y}{g}+\frac{z}{h}=0$.

(v) If S is inscribed to the $\triangle A_0B_0C_0$, the pairs of tangents from A_0, B_0, C_0 are $yz=0$, $zx=0$, $xy=0$ respectively; whence from (ii), $A=B=C=0$, so that $f^2=bc$, $g^2=ca$, and $h^2=ab$. Then, altering the notation by writing a^2, b^2, c^2 for a, b, c, the equation becomes

$$S \equiv a^2x^2+b^2y^2+c^2z^2 \pm 2bcyz \pm 2cazx \pm 2abxy=0.$$

To exclude the case of a coincident pair of lines, we take for the equation of an inscribed conic the form

$$S \equiv a^2x^2+b^2y^2+c^2z^2-2bcyz-2cazx-2abxy=0.$$

This at once yields the equivalent forms

$$\left.\begin{aligned} S &\equiv (-ax+by+cz)^2-4bcyz \\ &\equiv (ax-by+cz)^2-4cazx \\ &\equiv (ax+by-cz)^2-4abxy \end{aligned}\right\}$$

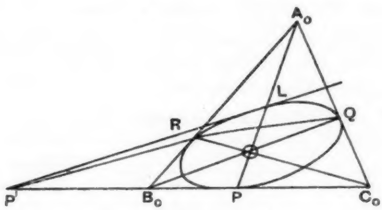

FIG. 1.

Then comparing with $w^2+kuv=0$, and using the obvious notation of the diagram the equations of QR, RP, PQ are $-ax+by+cz=0$, $ax-by+cz=0$, and $ax+by-cz=0$ respectively. Also P', Q', R' are ($x=0$, $by+cz=0$), etc., and thus P', Q', R' are collinear on the line $ax+by+cz=0$.
Again, we have immediately

$$\left.\begin{aligned} S &\equiv (by-cz)^2+ax(ax-2by-2cz) \\ &\equiv (cz-ax)^2+by(by-2cz-2ax) \\ &\equiv (ax-by)^2+cz(cz-2ax-2by); \end{aligned}\right\}$$

whence A_0P is the line $by-cz=0$, and the tangent at L is $ax-2by-2cz=0$, with similar results from the second and third forms.

Thus, A_0P, B_0Q, C_0R concur in the point O ($ax=by=cz$).

Also for the meet of the tangent at L with B_0C_0, we have ($x=0$, $by+cz=0$), which is the point P' previously found. Thus the tangent at L and QR meet on B_0C_0, the tangent at M and RP meet on C_0A_0, and the tangent at N and PQ meet on A_0B_0: and the three points so determined are collinear.

It is immediately verified that the line $P'Q'R'$ is the polar of O for the conic.

B. Let S meet the sides of the $\triangle A_0B_0C_0$, in the pairs of points P_1 and P_2, Q_1 and Q_2, R_1 and R_2.

Let $S \equiv \lambda x^2+f(xyz)$.

Then if $f(xyz)=0$ is a line-pair, it will represent the tangents at P_1 and P_2, by comparison with the type $w^2+kuv=0$, and clearly

$$f(xyz) \equiv (a-\lambda)x^2+by^2+cz^2+2fyz+2gzv+2hxy=0.$$

The condition for a line-pair is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0,$$

giving at once $\lambda = \dfrac{\Delta}{A}$.

Thus the pair of tangents at P_1 and P_2 is

$$\left(a - \frac{\Delta}{A}\right)x^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0, \quad \text{or} \quad S - \frac{\Delta}{A}x^2 = 0.$$

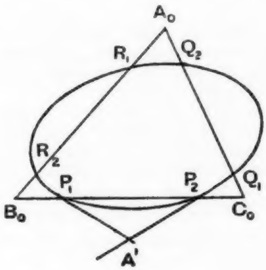

FIG. 2.

Similarly the pairs of tangents at Q_1 and Q_2, R_1 and R_2, are

$$S - \frac{\Delta}{B}y^2 = 0, \quad S - \frac{\Delta}{C}z^2 = 0 \quad \text{respectively.}$$

The points A', B', C', the respective poles of B_0C_0, C_0A_0, A_0B_0, may be obtained from these line-pairs, as in A (i), and it may then be proved that A_0A', B_0B', C_0C' are concurrent. This, however, appears more simply in line co-ordinates, as will now be explained.

II. Homogeneous Line Co-ordinates.

If (l, m, n) are the line co-ordinates which correspond to the point co-ordinates used above, and $\Sigma = 0$ is the equation of the same conic in these co-ordinates, then

$$\Sigma \equiv Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm.$$

The typical forms used are

$$\phi_2 + k\alpha\beta, = 0, \quad \phi_2 + k\alpha^2 = 0, \quad \gamma^2 + k\alpha\beta = 0, \quad \gamma\delta + k\alpha\beta = 0,$$

where ϕ_2 is quadratic, and $\alpha, \beta, \gamma, \delta$ are linear in l, m, n. The interpretation of these is sufficiently obvious.

A. We have

$$A\Sigma \equiv A^2l^2 + ABm^2 + ACn^2 + 2AFmn + 2AGnl + 2AHlm$$

$$\equiv (Al + Hm + Gn)^2 + m^2(AB - H^2) + n^2(AC - G^2) + 2mn(AF - GH)$$

or

$$\left.\begin{aligned} A\Sigma &\equiv (Al + Hm + Gn)^2 + \Delta(cm^2 - 2fmn + bn^2), \\ B\Sigma &\equiv (Hl + Bm + Fn)^2 + \Delta(an^2 - 2gnl + cl^2), \\ C\Sigma &\equiv (Gl + Fm + Cn)^2 + \Delta(bl^2 - 2hlm + am^2). \end{aligned}\right\}$$

and similarly

Comparing with the type $\gamma^2+k\alpha\beta=0$, we have the following conclusions:

(i) The equations of A', B', C', the poles of B_0C_0, C_0A_0, A_0B_0 are respectively

$$Al+Hm+Gn=0, \quad Hl+Bm+Fn=0, \quad Gl+Fm+Cn=0,$$

or
$$\frac{\partial\Sigma}{\partial l}=0, \quad \frac{\partial\Sigma}{\partial m}=0, \quad \frac{\partial\Sigma}{\partial n}=0.$$

Hence, also, the point co-ordinates of A', B', C' are $(A:H:G)$, $(H:B:F)$, and $(G:F:C)$ respectively; the lines A_0A', B_0B', C_0C' have point equations $\frac{y}{G}=\frac{z}{H}$, $\frac{z}{H}=\frac{x}{F}$, and $\frac{x}{F}=\frac{y}{G}$ respectively, and are thus concurrent in the point $(F:G:H)$.

(ii) If the conic Σ meets the sides of the $\triangle A_0B_0C_0$ in P_1 and P_2, Q_1 and Q_2, R_1 and R_2 respectively, the equations of the point pairs P_1 and P_2, Q_1 and Q_2, R_1 and R_2 are respectively $cm^2-2fmn+bn^2=0$, $an^2-2gnl+cl^2=0$, and $bl^2-2hlm+am^2=0$.

Thus the co-ordinates of A_0P_1, A_0P_2 are $(0, m_1, n_1)$, $(0, m_2, n_2)$, where $m_1:n_1$, and $m_2:n_2$ are the roots of the first of these equations, with similar results for B_0Q_1, B_0Q_2, and C_0R_1, C_0R_2. Then it is at once seen that these six lines touch the conic $bcl^2+cam^2+abn^2-2afmn-2bgnl-2chlm=0$. (On putting $l=0$, $m=0$, $n=0$ in turn, to find co-ordinates of pairs of tangents from A_0, B_0, C_0 respectively, we obtain the three equations above.)

(iii) If Σ has $A_0B_0C_0$ for a self-polar triangle, poles of B_0C_0, C_0A_0, A_0B_0 are $l=0$, $m=0$, $n=0$ respectively. Thus, from (i), $F=G=H=0$, and we have the form $\Sigma\equiv Al^2+Bm^2+Cn^2$.

(iv) If Σ is inscribed in the $\triangle A_0B_0C_0$, we have $P_1\equiv P_2$, $Q_1\equiv Q_2$, $R_1\equiv R_2$, whence, from (ii), $bc-f^2=ca-g^2=ab-h^2=0$, or $A=B=C=0$;

$$\therefore \ \Sigma\equiv Fmn+Gnl+Hlm.$$

We may write this in the forms

$$\left.\begin{aligned}\Sigma&\equiv Fmn+GHl\left(\frac{n}{H}+\frac{m}{G}\right)\\ &\equiv Gnl+HFm\left(\frac{l}{F}+\frac{n}{H}\right)\\ &\equiv Hlm+FGn\left(\frac{m}{G}+\frac{n}{H}\right).\end{aligned}\right\}$$

Then, comparing with the type $\gamma\delta+k\alpha\beta=0$, we see that, if P, Q, R be the points of contact with B_0C_0, C_0A_0, A_0B_0 respectively, the equations of the points P, Q, R are $\frac{m}{G}+\frac{n}{H}=0$, $\frac{n}{H}+\frac{l}{F}=0$, $\frac{l}{F}+\frac{m}{G}=0$ respectively. The co-ordinates of these points are then $(0, G^{-1}, H^{-1})$, etc., the point equations of A_0P, B_0Q, C_0R are $Gy=Hz$, etc., and these three lines concur in the point $(F^{-1}:G^{-1}:H^{-1})$.

(v) If Σ is circumscribed to the $\triangle A_0B_0C_0$, then P_1 and P_2 are B and C, etc., whence, from (ii), $a=b=c=0$, so that

$$BC-F^2\equiv a\Delta=0, \quad CA-G^2\equiv b\Delta=0, \quad AB-H^2\equiv c\Delta=0;$$

$$\therefore \ \Sigma\equiv Al^2+Bm^2+Cn^2\pm2\sqrt{BC}mn\pm2\sqrt{CA}nl\pm2\sqrt{AB}lm.$$

Then writing A^2, B^2, C^2 for A, B, C, and choosing signs to avoid the coincident point-pair, we may take

$$\Sigma\equiv A^2l^2+B^2m^2+C^2n^2-2BCmn-2CAnl-2ABlm.$$

This may be re-arranged in the forms

$$\left.\begin{aligned}\Sigma &\equiv(-Al+Bm+Cn)^2-4BCmn\\ &\equiv(Al-Bm+Cn)^2-4CAnl\\ &\equiv(Al+Bm-Cn)^2-4ABlm.\end{aligned}\right\}$$

Comparing with the form $\gamma^2+k\alpha\beta=0$, and using the obvious notation of the diagram, we see that A', B', C' have equations $-Al+Bm+Cn=0$,

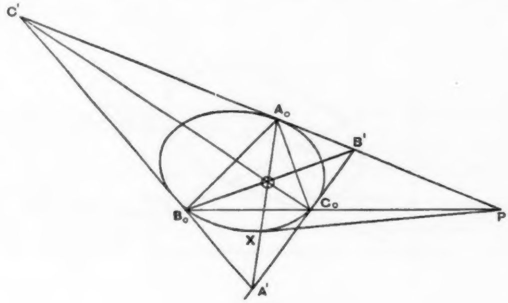

FIG. 3.

$Al-Bm+Cn=0$, $Al+Bm-Cn=0$ respectively, the lines A_0A', B_0B', C_0C have co-ordinates $(0:C:-B)$, $(-C:0:A)$, $(B:-A:0)$ respectively, and concur in the point with the equation $Al+Bm+Cn=0$.

Again, we may write

$$\left.\begin{aligned}\Sigma &\equiv(Bm-Cn)^2+Al(Al-2Bm-2Cn)\\ &\equiv(Cn-Al)^2+Bm(Bm-2Cn-2Al)\\ &\equiv(Al-Bm)^2+Cn(Cn-2Al-2Bm).\end{aligned}\right\}$$

Comparing with $\gamma^2+k\alpha\beta=0$, P, Q, R are the points $Bm-Cn=0$, $Cn-Al=0$, $Al-Bm=0$, and lie on the line $Al=Bm=Cn$, whose co-ordinates are (A^{-1}, B^{-1}, C^{-1}). Further, if X, Y, Z are the points of contact of the second tangent from P, Q, R respectively, their equations are $Al-2Bm-2Cn=0$, etc., and from the above values for the co-ordinates of A_0A', etc., we see that X lies on A_0A', Y on B_0B', and Z on C_0C'.

III. Non-homogeneous (Cartesian) Co-ordinates.

This case is considered last, as it is the degenerate case of the preceding, which is obtained by passing one side of the triangle of reference to infinity. On making the appropriate changes in geometrical interpretation, the results may be deduced from those already given. We give a separate discussion, as students will usually encounter this case first.

The point equation is

$$S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0;$$

whence

$$\begin{aligned}aS &\equiv(ax+hy+g)^2+Cy^2-2Fy+B,\\ bS &\equiv(hx+by+f)^2+Cx^2-2Gx+A,\\ cS &\equiv(gx+fy+c)^2+Bx^2-2Hxy+Ay^2.\end{aligned}$$

Comparing with the type $w^2 + kuv = 0$, we have the following results:

(i) The pair of tangents parallel to Ox is $Cy^2 - 2Fy + B = 0$, and their chord of contact, or the diameter conjugate to Ox, is $ax + hy + g = 0$.

(ii) The pair of tangents parallel to Oy is $Cx^2 - 2Gx + A = 0$, and their chord of contact, or the diameter conjugate to Oy, is $hx + by + f = 0$.

(iii) The pair of tangents from the origin is $Bx^2 - 2Hxy + Ay^2 = 0$, and the polar of the origin is $gx + fy + c = 0$.

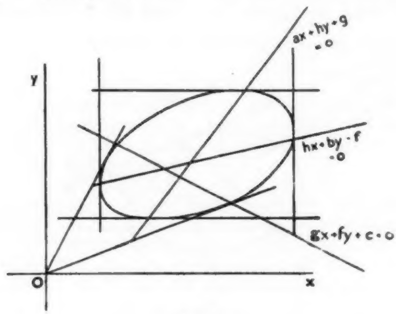

FIG. 4.

Re-arranging S, as in I. B., we have

$$\left.\begin{aligned} S &\equiv \frac{\Delta}{A}x^2 + \left(S - \frac{\Delta}{A}x^2\right) \\ &\equiv \frac{\Delta}{B}y^2 + \left(S - \frac{\Delta}{B}y^2\right) \\ &\equiv \frac{\Delta}{C} \quad + \left(S - \frac{\Delta}{C}\right), \end{aligned}\right\} \text{where in each case the latter term is a product of linear factors;}$$

whence $S - \frac{\Delta}{A}x^2 = 0$, $S - \frac{\Delta}{B}y^2 = 0$ are the respective pairs of tangents at the meets of the conic with the co-ordinate axes. Also $S - \frac{\Delta}{C} = 0$ will represent the tangents at the meets of S with the line infinity, or the asymptotes.

If $\Sigma \equiv Al^2 + 2Hlm + Bm^2 + 2Gl + 2Fm + C = 0$ be the corresponding equation in line co-ordinates (l, m), we have

$$\begin{aligned} A\Sigma &\equiv (Al + Hm + G)^2 + (AB - H^2)m^2 + (AC - G^2) + 2(AF - GH)m \\ &\equiv (Al + Hm + G)^2 + \Delta(cm^2 - 2fm + b), \end{aligned}$$

and similarly
$$\begin{aligned} B\Sigma &\equiv (Hl + Bm + F)^2 + \Delta(cl^2 - 2gl + a), \\ C\Sigma &\equiv (Gl + Fm + C)^2 + \Delta(bl^2 - 2hlm + am); \end{aligned}$$

whence, comparing with the form $\gamma^2 + k\alpha\beta = 0$, we see that

(i) Pole of Ox is $Hl + Bm + F = 0$, pole of Oy is $Al + Hm + G = 0$, and pole of line infinity, or centre of conic, is $Gl + Fm + C = 0$. The co-ordinates of these respective points are $\left(\frac{H}{F}, \frac{B}{F}\right)$, $\left(\frac{A}{G}, \frac{H}{G}\right)$, and $\left(\frac{G}{C}, \frac{F}{C}\right)$.

(ii) The pairs of meets of the conic with Ox, Oy, and the line infinity, are the line-pairs $cl^2 - 2gl + a = 0$, $cm^2 - 2fur + b = 0$, and $bl^2 - 2hlm + am^2 = 0$ respectively.

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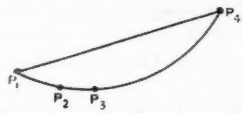
A FORMULA IN INEQUALITIES.

BY H. LOB, M.A.

IF particles of positive mass lie on a continuous curve whose concavity is always upwards, their centroid clearly lies above the curve and below the chord joining the extreme particles.



This obvious property gives a simple formula which includes several of the standard inequalities of the Algebra text-books.

Let $(x_1, y_1)(x_2, y_2) \dots (x_n, y_n)$ be the co-ords. of points $P_1, P_2, \dots P_n$ lying on a curve $y=f(x)$, and $a_1, a_2, \dots a_n$ the masses (positive) of particles placed at these points.

Let $f(x)$ be finite, continuous and single-valued in the interval x_1 to x_n. Then if the curve is always concave upwards in this interval, *i.e.* if $f''(x)$ is positive for all values of x from x_1 to x_n, we get

$$\bar{y} > f(\bar{x})$$
$$< \frac{(x_n - \bar{x})y_1 + (\bar{x} - x_1)y_n}{x_n - x_1},$$

where
$$\bar{x} = \frac{\Sigma ax}{\Sigma a}, \quad \bar{y} = \frac{\Sigma ay}{\Sigma a}.$$

In what follows, we shall only consider the first inequality. Thus

$$\frac{a_1 f(x_1) + a_2 f(x_2) + \dots + a_n f(x_n)}{a_1 + a_2 + \dots + a_n} > f\left(\frac{a_1 x_1 + a_2 x_2 + \dots + a_n x_n}{a_1 + a_2 + \dots + a_n}\right). \text{ ....(A)}$$

It is clear that the centroid property above stated holds equally good when the curve consists wholly or in part of straight lines, *e.g.* it may be a convex polygon, provided that $f'(x)$ never diminishes in the interval x_1 to x_n.

By taking the corresponding property for a curve which is convex upwards, we get, if $f''(x)$ is always negative in the interval x_1 to x_n,

$$\bar{y} < f(\bar{x}),$$

i.e.
$$\frac{a_1 f(x_1) + a_2 f(x_2) + \dots + a_n f(x_n)}{a_1 + a_2 + \dots + a_n} < f\left(\frac{a_1 x_1 + a_2 x_2 + \dots + a_n x_n}{a_1 + a_2 + \dots + a_n}\right). \text{ ....(A}')$$

The following examples show the scope of the formula :

(1) $y = x^2$.

The curve is concave upwards for all values of x.

$$\therefore \frac{a_1 x_1^2 + a_2 x_2^2 + \dots + a_n x_n^2}{a_1 + a_2 + \dots + a_n} > \left(\frac{a_1 x_1 + a_2 x_2 + \dots + a_n x_n}{a_1 + a_2 + \dots + a_n}\right)^2$$

for all values of $x_1, \dots x_n$.

(2) $y = x^3$.

The curve is concave upwards for all positive values of x;

$$\therefore \frac{a_1 x_1^3 + a_2 x_2^3 + \dots + a_n x_n^3}{a_1 + a_2 + \dots + a_n} > \left(\frac{a_1 x_1 + a_2 x_2 + \dots + a_n x_n}{a_1 + a_2 + \dots + a_n}\right)^3$$

for positive values of $x_1, x_2, \dots x_n$.

(3) $y = x^m$ (where m is any number).

We shall take x positive, and for fractional values of m we shall take the positive real value of x^m. We then get a continuous curve with one branch, which is concave upwards or convex upwards, according as $m(m-1)x^{m-2}$ is positive or negative.

Hence the curve is concave upwards if m does not lie between 0 and 1, and convex upwards if m lies between 0 and 1.

$$\therefore \quad \frac{a_1 x_1{}^m + a_2 x_2{}^m + \ldots + a_n x_n{}^m}{a_1 + a_2 + \ldots + a_n} \gtrless \left(\frac{a_1 x_1 + a_2 x_2 + \ldots + a_n x_n}{a_1 + a_2 + \ldots + a_n}\right)^m,$$

according as m does not or does lie between 0 and 1.

(4) $y = \sin x$.

The curve is convex upwards for all values of x between 0° and 180°.

$$\therefore \quad \frac{a_1 \sin x_1 + a_2 \sin x_2 + \ldots + a_n \sin x_n}{a_1 + a_2 + \ldots + a_n} < \sin\left(\frac{a_1 x_1 + a_2 x_2 + \ldots + a_n x_n}{a_1 + a_2 + \ldots + a_n}\right)$$

for all values of the x's between 0° and 180°.

In particular, $\dfrac{\sin A + \sin B + \sin C}{3} < \dfrac{\sqrt{3}}{2}$ for any triangle ABC which is not equilateral.

(5) $y = \sin^m x$ (where m is a positive number >1).

$$f''(x) = m \sin^{m-2} x \left[(m-1) - m \sin^2 x\right].$$

Hence if $x_1, x_2, \ldots x_n$ are all between 0 and $\sin^{-1}\sqrt{\dfrac{m-1}{m}}$, we have

$$\frac{a_1 \sin^m x_1 + a_2 \sin^m x_2 + \ldots + a_n \sin^m x_n}{a_1 + a_2 + \ldots + a_n} > \sin^m\left(\frac{a_1 x_1 + a_2 x_2 + \ldots + a_n x_n}{a_1 + a_2 + \ldots + a_n}\right).$$

In particular, if x_1, x_2, x_3 are all between 0° and 60°, then

$$\frac{\sin^4 x_1 + \sin^4 x_2 + \sin^4 x_3}{3} > \sin^4 \frac{x_1 + x_2 + x_3}{3}.$$

(6) $y = c^x$, where c is positive.

Here, again, we take the positive real value of c^x.

The curve is then continuous, single-branched and concave upwards for all values of x.

Hence $$\frac{a_1 c^{x_1} + a_2 c^{x_2} + \ldots + a_n c^{x_n}}{a_1 + a_2 + \ldots + a_n} > c^{\frac{a_1 x_1 + a_2 x_2 + \ldots + a_n x_n}{a_1 + a_2 + \ldots + a_n}}.$$

Call $c^{x_1} \equiv \alpha_1, \quad c^{x_2} \equiv \alpha_2, \ldots$.

Then we get

$$\frac{a_1 \alpha_1 + a_2 \alpha_2 + \ldots + a_n \alpha_n}{a_1 + a_2 + \ldots + a_n} > \left(\alpha_1^{a_1} \alpha_2^{a_2} \ldots \alpha_n^{a_n}\right)^{\frac{1}{a_1 + a_2 + \ldots + a_n}}$$

for all positive values of the a's and α's.

In particular, if $a_1 = a_2 = \ldots = a_n = 1$, we have

$$\frac{\alpha_1 + \alpha_2 + \ldots + \alpha_n}{n} > \sqrt[n]{\alpha_1 \alpha_2 \ldots \alpha_n}$$

for all positive values of the α's.

The formula may be extended to functions of two variables by taking particles of positive mass on a synclastic surface. If $z = f(x, y)$ is the equation of a surface, the conditions that it should be concave upwards are

$\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$, both positive, and $\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 < \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2}$, while the conditions that it should be convex upwards are $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$, both negative, and

$$\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 < \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2}$$

("upwards" of course now meaning the positive direction of the z-axis).

And $\bar{z} \gtrless f(\bar{x}, \bar{y})$, according as the surface is concave upwards or convex upwards, ...(B)

where $$z \equiv \frac{\Sigma az}{\Sigma a}, \quad \bar{x} \equiv \frac{\Sigma ax}{\Sigma a}, \quad \bar{y} \equiv \frac{\Sigma ay}{\Sigma a}.$$

Thus, take $z = \cos x \cos y$.

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2} = -\cos x \cos y; \quad \frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y.$$

Hence if the x's and y's are all between 0° and 45°, we have

$$\frac{a_1 \cos x_1 \cos y_1 + a_2 \cos x_2 \cos y_2 + \ldots + a_n \cos x_n \cos y_n}{a_1 + a_2 + \ldots + a_n}$$
$$< \cos \frac{a_1 x_1 + a_2 x_2 + \ldots + a_n x_n}{a_1 + a_2 + \ldots + a_n} \cos \frac{a_1 y_1 + a_2 y_2 + \ldots + a_n y_n}{a_1 + a_2 + \ldots + a_n}.$$

Definite Integrals. We may take the particles continuously along the curve, or, what comes to the same thing, we may imagine the curve and particles to be replaced by a wire of variable density.

Then if the a's are all equal to 1, the formula (A) becomes

$$\frac{\int_{x_1}^{x_n} f(x)\,dx}{x_n - x_1} > f\left(\frac{x_1 + x_n}{|2}\right).$$

If a_r is a function of x_r, say $a_r \equiv \phi(x_r)$, where $\phi(x)$ is positive for all values of x in the interval x_1 to x_n, the formula becomes

$$\frac{\int_{x_1}^{x_n} \phi(x) f(x)\,dx}{\int_{x_1}^{x_n} \phi(x)\,dx} > f(X), \quad \text{where} \quad X = \frac{\int_{x_1}^{x_n} x\phi(x)\,dx}{\int_{x_1}^{x_n} \phi(x)\,dx},$$

and similarly, of course, for the formula (A′).

The formula (B) lends itself, in the same way, to double integration.

Manchester Grammar School. H. LOB.

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205. If any of our London mathematicians be unwilling or unable to write Latine, let them write English, or the catholicke language [a, b, c, + − &c.]. . . . John Pell to John Leak. Aug. 7, 1645.

206. The 3 vexations of scientificall mortals . . . to wit, the squaringe of the circle, the dublinge of the cube, and the philosopher's stoone. W. Lower to T. Harriot, 1611.

207. Confidents du Très Haut, substances éternelles,
Qui brûlez de ses feux, qui couvrez de vos ailes
Le trône où votre maître est assis parmi vous,
Parlez : du grand Newton n'étiez-vous point jaloux ?—Voltaire.

SIMILARITY; OR LINE UPON LINE, PRINCIPAL UPON PRINCIPLE.

Period. 192 ?.

Scene. A mathematical class-room. The usual audience of Eager (?) Seekers after knowledge. The lecturer introduces the reformed methods of teaching elementary Geometry.

Lecturer. After the course just finished everyone here should be completely convinced that 2 rectilineal figures are geometrically similar when they are equiangular to one another and have the sides about the equal angles proportional, taken in order. Are you all agreed about that ?

Chorus. Yes, sir.

Lecturer. Very well. I shall now try to make you understand what is meant by logical deduction and how one geometrical fact leads on inevitably to other new facts; but of course, bricks cannot be made without straw, nor can geometrical facts be deduced without some geometrical straws; assumptions I prefer to call them, or—perhaps better—principles.

1st E.S. (whose attention has not yet wandered). Please, sir. Can't we find out these new facts by drawing and measuring ? We used to do it that way.

Lecturer (after pulverising the unlucky interrupter). The principle I want you all to get hold of is as follows: I will write it on the board, "On a given straight line it is always possible to draw a rectilineal figure similar to a given figure." For example, given a square and a line of any length whatever, then another square can be drawn on the given line, and of course the two squares are similar. In fact, all we are saying is that it is possible to draw figures to different scales anywhere we like.

2nd E.S. Oh, sir, then we *can* use our instruments after all.

Lecturer (slightly ruffled). Don't say "can" when you mean "may." There isn't very much you *can* do properly yet. (Collapse of 2nd E.S.)

Lecturer (continuing with rising enthusiasm). The principle I have written up is called "The Principle of Similarity." We shall find it makes everything simple and straightforward. Let us begin with triangles.

. . . .

The lecturer then enunciates the three fundamental theorems on similar triangles and proceeds to their demonstration. (Reference, *Math. Gazette*, May 1922, p. 70.)

Lecturer. Now then, it is your turn. Tell me first what the Principle of Similarity is.

A confused murmur arises which eventually merges into a chorus repeating correctly the Principle asked for.

Lecturer. We are not getting on as fast as I could wish, so perhaps for the rest of the hour you had better take your instruments and draw figures illustrating the principle. Take a triangle with sides 6, 8, 10 cm. as the given figure and a line 5 cm. long as the given line.

The lecturer sits down and reads the *Gazette* while the class set to work. A considerable interval, during which the only sounds are the shuffling of feet and the falling of instruments—and boxes—on the floor.

. . . .

Excited Voice. Please, sir.

Lecturer (sitting up). Well, what is it ?

E.V. Please, sir, isn't C fixed ?

Lecturer. I do not understand; you had better bring it up.

This is what the lecturer saw.

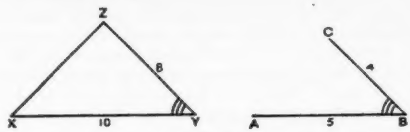

FIG. 1.

E.V. Please, sir, I made $\angle B = \angle Y$ and then made BC 4 cm. Isn't C fixed then, sir ?

Lecturer. Of course, C will be a fixed point if you did that.

E.V. But, sir, if it's fixed, then everything's fixed.

Lecturer. Yes, the triangle is completely determined.

E.V. But my triangles are similar, aren't they, sir ?

Lecturer (a little uneasy). Yes.

E.V. I'm afraid I forgot to mark the length of XZ, sir, but it doesn't seem to matter.

Lecturer (still more uneasy). No, it doesn't.

2nd E.V. Please, sir, may I bring this up ?

Lecturer (glad of the interruption). Yes, certainly.

Here is the second boy's diagram.

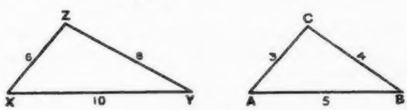

FIG. 2.

2nd E.V. Please, sir, I found C by drawing two arcs with centres A and B and radii 3 cm. and 4 cm. My triangles are similar, aren't they, sir ?

Lecturer. By the principle of similarity, you mean ?

2nd E.V. Yes, sir.

Lecturer (very slowly). I suppose they are.

3rd E.V. Please, sir, can I come up ?

Lecturer. Yes. You other two sit down.

Here is the third diagram—as the lecturer expected by now.

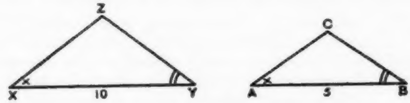

FIG. 3.

3rd E.V. I made $\angle A = \angle X$, sir, and $\angle B = \angle Y$, sir, and produced the lines to meet at C, sir. Oh ! I'm afraid I haven't marked the lengths of XZ and YZ, sir. But the triangles are similar, aren't they, sir ?

Lecturer (gloomily). It seems to be all right.

The clock strikes.

Lecturer (much relieved). All right, you can go now.

The class goes out, leaving the lecturer in thought. WEE SLIP.

Prof. Nunn remarks :—If the lecturer had not collapsed before the queries of the First Excited Voice he would have seen that the boy's difficulty pointed

straight to a valuable simplification of the proof given in my article in the *Gazette* for May. By the (assumed) Principle of Similarity, there can be drawn on the upper side of AB a triangle having its sides one-half of the corresponding sides of XYZ and its angles equal to the corresponding angles of that triangle. Now when the construction has gone as far as is shown in Fig. 1, it is clear that there is only one triangle in which AB and BC are one-half of XY and YZ and $\angle B = \angle Y$; for, as the Voice said, the point C is fixed. Thus ABC must be the triangle similar, by hypothesis, to XYZ. It follows, by an obvious generalisation, that two triangles must be similar if they have a pair of corresponding angles equal and if the corresponding sides about those angles are in the same ratio. *Mutatis mutandis*, the argument applies equally well to the other cases.

Why did the lecturer miss this opportunity of improving my more cumbrous proof? Perhaps because he did not see that, apart from a principle of similarity, proved as in Euclid or assumed as in my scheme, there is no guarantee whatever that the completed triangle ABC in Fig. 1 will be equiangular to XYZ. If the figures were drawn upon a sphere, in the manner explained in the article, the angles at C and A would *not* be equal to those at Z and X, and CA would *not* be one-half of ZX.

I fear, however, that these pedantic remarks spoil what is meant to be the point of a *jeu d'esprit*. T. P. NUNN.

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208. A recent effort:—$\log x = 5$; $\therefore\ x = \frac{5}{\log}$.

209. Statues, vitraux, arabesques, dentelures, chapiteaux, bas-reliefs, elle combine toutes ces imaginations selon le logarithme qui lui convient. Victor Hugo: *Notre Dame de Paris*.

210. I found the house [Hawarden Castle] full of young people. . . . The young man who strikes me most is Strutt, the senior wrangler [Lord Rayleigh, Senr. Wrangler 1865]. A. Hayward to Lady Waldegrave. Dec. 9, 1870.

211. After six fits of a quartan ague with which it pleased God to visite him, died my deare son Richard . . . 5 yeares and 3 days old onely. . . . He had a wonderful disposition to mathematics, having by heart divers propositions of Euclid that were read to him in play, and he would make lines and demonstrate them.—*Evelyn's Diary*, Jan. 27, 1658.

212. In common with all mathematicians of my acquaintance, I am no prompt debater, no acute logician, no clear expositor, but begin in hesitation and finish in confusion.—Landor, *Pericles and Aspasia*.

213. A recondite treatise on *Trigonometry* was condemned because the critics imagined it contained heretical opinions concerning the Trinity, and another work on *Insects*, because a secret attack on the Jesuits. Well might Galileo exclaim: "Are these my judges?" But the ways of censors are past anticipation. Is there much between the men who deleted "the Captains and the Kings depart," or ". . . entered into rest," and the Russians who destroyed a volume on *The Revolutions of the Heavenly Bodies*?

214. Gavarni is rather eccentric. He has taken up higher mathematics and insists on contradicting Newton. He has written a memoir on "Speed within Speed."—*Taine*. [? Gavarni the artist.]

215. Mr. Mill presents his compliments to Mr. Murray, and begs he will accept his cordial thanks for the loan of the *Sanscrit Algebra*. . . .—James Mill to John Murray, 1 Queen's Square, Westminster, July 25, 1817.

216. John Keble to his brother, Aug. 1819: If you want to sleep, I recommend the *Greek Metres*, and if you want a headache, I recommend Maclaurin's *Account of Newton's Discoveries*.

MATHEMATICAL NOTES.

694. [K¹. 9. b.; X. 7.] *A Geometrical Construction of the Regular Heptagon (by means of the Elliptic Compasses).*

Describe a circle (centre C, radius $=1$) ADB (Fig. 1).
Draw diameter AB and ordinate DC.
Take point E on circumference, so that angle $ACE=60°$.
From E draw EF perpendicular to CD.
Take G so that $GF=EF/3=AC/6$.

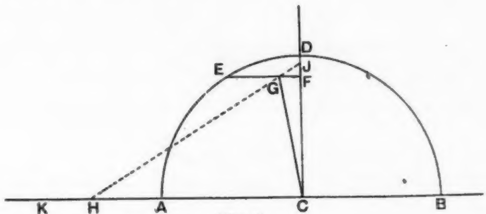

FIG. 1.

Join CG, and now apply the elliptic compasses so that the right-angled frame coincides with ACD, and with the moving limb HJ adjusted to a length equal to $2.CG$. Slide the limb till it passes through G, without being bisected there. Take HK equal to EG $(=AC/3)$.

Now, by construction we have obtained that

$$\text{angle } GCB = 3 \text{ times angle } GHC,$$

that $$2\cos GCB = -1/\sqrt{7},$$

that $$CG=\sqrt{7}/3;$$

so that calling $$2\cos GHC=a,$$

we have $$KC=(a\sqrt{7}+1)/3=s_1, \quad \text{.......(I)}$$

$$a^3-3a=-1/\sqrt{7}, \quad \text{.......(II)}$$

$$a^4=3a^2-a/\sqrt{7}. \quad \text{.......(III)}$$

Again, proceeding to define two other values, s_2 and s_3, from the conditions

$$s_1^2-2=s_2,$$

$$s_2^2-2=-s_3,$$

we get first that $$s_2=(7a^2+2a\sqrt{7}-17)/9, \quad \text{.......(IV)}$$

and then that $$-s_3=(49a^4+28a^3\sqrt{7}-210a^2-68a\sqrt{7}+127)/81,$$

which by (II) and (III) becomes

$$s_3=(7a^2-a\sqrt{7}-11)/9, \quad \text{.......(V)}$$

and further that similarly $s_3^2-2=-s_1$.

So that we have s_1 connected with the radius by a system of three equations (written most conveniently for the geometer):

$$s_1^2-1^2=1(1+s_2), \quad \text{.......(VI)}$$

$$s_2^2-1^2=1(1-s_3), \quad \text{.......(VII)}$$

$$1^2-s_3^2=1(s_1-1). \quad \text{.......(VIII)}$$

Taking (Fig. 2) any point O on the circle and describing another circle (centre O) through C, we place the chord ON ($=s_1$) cutting the new circle in P.

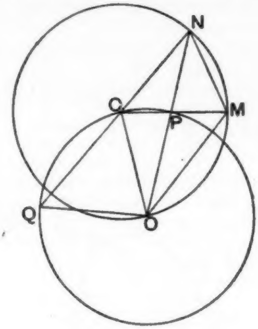

FIG. 2.

Join CO, NC and CP, and produce to cut circles in Q and M. Join MN.

Then because $ON=s_1$,
we have by (VI) that $CQ=s_2$,
and by (VIII) that $CP=s_3$.
And because $CP=s_3$,
we have by (VII) that $OM=s_2$.

$\therefore$ $OM=CQ$, and $\therefore$ CM is parallel to QO;

$\therefore$ angle $NCM=$ angle $CQO=QCO=2$ angle CNO or COP;

$\therefore$ angle $CPO=3$ times angle COP;

and $\therefore$ CP is a side of the regular inscribed 14-gon
and MN „ „ „ Heptagon.

[It is perhaps interesting to note that if we take a point T on the diameter ACB of a circle (of unit radius), so that $AT(>1):1::CT:TB$, we obtain the *Pentagon*; but if we take T so that $AT:1::CT^2:TB^2$, we get the *Heptagon*. In this latter case, $CT=3/(a\sqrt{7}+1)$, being in fact the reciprocal of our s_1, and the construction affords a neat exercise.]

85 Wilbury Crescent, Hove, Sussex, C. H. CHEPMELL.
August 15, 1923.

695. [C[1]. 16. b.] *The Diameter of a Pedal Circle.*

"Wanted, a simple proof (geometrical) that the polar circle of a triangle cuts orthogonally the director circle of any inscribed conic—a proof which does not approach the triangle by way of the quadrilateral."

This request, which appeared over the name of one of the most fertile of geometers, in the *Gazette* for December, 1906 (vol. iii, p. 406), was the cause of several notes in later numbers. In one of these, the propounder pointed out that just such a proof as he had in mind was given in the second number of the *Gazette* (July, 1894) by Mr. E. P. Rouse; this proof was reproduced "for the benefit of a younger generation" (vol. iv, p. 311).

A feature of the proposition to which attention was not drawn at that time is, that the reference to an inscribed conic is wholly superfluous. If the conic,

inscribed in a triangle ABC whose orthocentre is H, has foci S, T, centre Q, and semi-axes α, β, the condition of orthogonality is

$$\alpha^2+\beta^2+\rho^2=QH^2$$
$$=\tfrac{1}{2}(SH^2+TH^2)-(\alpha^2-\beta^2),$$

that is $$4\alpha^2=SH^2+TH^2+2\rho^2, \qquad \text{.................................(1)}$$

where ρ is the radius of the polar circle, and this can be regarded as a formula giving the diameter 2α of the pedal circle which has a given pair of isogonal conjugates S, T for its poles. From this point of view, there is interest in a simple proof which does not introduce the conic at all; the proof which follows may be compared with a proof given by Prof. A. C. Dixon (vol. iv, p. 317) in response to the original request.

Let d, e, f be the projections of S on the sides of the triangle, U, V, W the images of S in these sides. Then TW, being $2Qf$, is equal to the diameter of the pedal circle of S and T. Also $\angle BAW=\angle SAB=\angle CAT$, and therefore $\angle TAW=A$. Thus, since AS, AW have the same length, the diameter δ which is to be found is given by

$$\delta^2=AS^2+AT^2-2AW.AT\cos A. \qquad \text{...........................(2)}$$

Now H, the orthocentre, is the centroid of ABC for multipliers $\tan A$, $\tan B$, $\tan C$. Hence, for any point P,

$$\Sigma AP^2\tan A=PH^2\Sigma\tan A+\Sigma AH^2\tan A.$$

On multiplying (2) by $\tan A$, we introduce the expression $2AW.AT\sin A$, and this, being equal to $4\Delta TAW$ and also to $4\Delta VAT$, is twice the area of the quadrilateral $TVAW$. Thus we have

$$(\delta^2-SH^2-TH^2)\Sigma\tan A=2\Sigma AH^2\tan A-2\Sigma TVAW. \qquad \text{..........(3)}$$

But $\Sigma TVAW$, the area of the hexagon $VAWBUC$, is $2\Delta ABC$. It follows that $\delta^2-SH^2-TH^2$ has a value which, being expressible as

$$(2\Sigma AH^2\tan A-4\Delta ABC)/\Sigma\tan A,$$

is the same for all pedal circles, and since the nine-points circle, whose diameter is the circumradius R, is itself a pedal circle and has for its poles the circumcentre O and the orthocentre H, the constant value of $\delta^2-SH^2-TH^2$ is R^2-OH^2, the negative of the power of H for the circumcircle.

Direct proof of the identity

$$(R^2-OH^2)\Sigma\tan A=2\Sigma AH^2\tan A-4\Delta ABC$$

is unnecessary, as also is recognition that the individual area ΔTAW is equal to that of the quadrilateral $TeAf$. It will be noticed that the result

$$\delta^2=SH^2+TH^2+R^2-OH^2,$$

and the proof given, both remain valid whether or not the triangle has a real polar circle, and whether or not the in-conic of which S and T are foci has a real director circle.

E. H. NEVILLE.

696. [K^1. 6. a.] Will you kindly allow me a little space to comment on a Note by Mr. G. J. Lidstone in the *Gazette*, p. 345, on account both of the importance of the point he raises, and of his implied criticism of the way in which I deal with it, in my *Plane Geometry*, his kindly reference to which I much appreciate.

Put briefly, his contention appears to be that for the concurrence of the three straight lines

$$a_rx+b_ry+c_r=0 \quad (r=1, 2, 3),$$

the condition $\{a_1b_2c_3\}=0$, is *necessary*, but *not sufficient*, for he says that the further conditions C_1, C_2, C_3, all $\neq 0$, are needed to ensure an accessible point of concurrence.

Surely these further conditions merely impose a limitation on the *position* of the point of concurrence. The position of parallelism of the three lines merely corresponds to a particular position of their common point, and should not be excluded from the property of concurrence. Failure to recognise this seems to strike at the very root of modern ideas of continuity in geometry.

The invalidity of the proposed distinction is seen immediately, if we recollect that a simple projection of the three parallel straight lines, for which the concurrence property is invariant, will bring their point of concurrence into an accessible position.

Thus the condition $\{a_1 b_2 c_3\} = 0$ is surely both *necessary* and *sufficient* for concurrence. In any particular case, the solution of two of the equations $a_r x + b_r y + c_r = 0$ will give the actual situation (accessible or otherwise) of the common point. L. B. BENNY.

[Prof. Neville remarks :—The idea that parallelism is a form of concurrence is modern only in the sense that it is not classical. In Kepler's *Ad Vitellionem Paralipomena, Quibus Astronomiæ Pars Optica Traditur*, 1604, occurs the sentence (Cap. IV, § 4 ; p. 93 of the original edition):

In Parabole [focus] unus . . . est intra sectionem, alter vel extra vel intra sectionem in axe fingendus est infinito interuallo à priore remotus, adèo ut [linea] educta . . . ex illo cæco foco in quodcunque punctum sectionis . . . sit axi . . . parallelos.

Pascal's famous *Essais pour les Coniques*, 1640, of which an annotated translation will shortly be published in the *Gazette*, begins as follows (we transcribe from the 1819 edition of the *Œuvres*, t. IV, p. 1):

Définition I. Quand plusieurs lignes droites concourent au même point, ou sont toutes parallèles entre elles : toutes ces lignes sont dites de *même ordre* ou de *même ordonnance ;* et la multitude de ces lignes, est dit *ordre de lignes*, ou *ordonnance de lignes*.]

697. [K[1]. 20. e.] *Note on a useful formula in Trigonometry.*

With data b, c, A,

$$\cot B = \frac{c}{b}\operatorname{cosec} A - \cot A.$$

For

$$\cot B = \frac{a \cos B}{a \sin B} = \frac{c - b\cos A}{b \sin A} = \frac{c}{b}\operatorname{cosec} A - \cot A.$$

A similar formula applied to the division of an angle A into two parts B and C whose sines shall have a given ratio, *e.g.* $\sin C = m \sin B$,

$$\cot B - \cot A = \frac{\sin(A - B)}{\sin A \sin B} = m \operatorname{cosec} A,$$

$$\cot B = m \operatorname{cosec} A + \cot A.$$

E.g. The resultant R of two forces P and Q acting at an angle θ, makes with the line of action of P an angle whose cotangent $= \frac{P}{Q}\operatorname{cosec}\theta + \cot\theta$.

E. M. LANGLEY.

REVIEWS.

Biomathematics: being the Principles of Mathematics for Students of Biological Science. By W. M. FELDMAN. Pp. 387. 21s. net. 1923. (Charles Griffin & Co.)

This book is an endeavour to present almost all the mathematical preliminaries necessary for the serious student at the meeting-point of physico- and bio-chemistry with physiology.

It must have been written with great energy and enthusiasm. Illuminating examples gleam from every page, and to one non-physiological reader, at any rate, they seem both appropriate and delightful.

Rather more than a quarter of the book consists of miscellaneous mathematics, preparatory to the middle half, which is on differential and integral calculus, with a chapter on differential equations. The remainder is chiefly concerned with bio-metrical and statistical applications.

Some adverse criticism must be made on the purely mathematical investigations. For example, very great stress is wisely laid at the beginning on the absurdity of working out results numerically to a degree of accuracy in excess of the accuracy of the data; but—physician, heal thyself! Again, the "exponential series" is obtained in three lines from the binomial expansion without comment. It would be an unpleasant and ungrateful task, considering the essential value of the book, to record further blemishes. The fact is, that, while it is admittedly undesirable to waste space on excessive precision in a work of this kind, the mathematical methods could be revised and pulled together with great advantage.

Most teachers in mathematics continually receive requests from workers in the biological field for a "useful book on the Calculus." There are many such for the mechanical engineer, but, as far as the present writer knows, this book fills a gap.

J. P. CLATWORTHY.

Our Debt to Greece and Rome. Mathematics. By PROF. D. E. SMITH. Pp. 164. 5s. 1923. (Harrap & Co.)

This is one of a library of about fifty volumes, the formation and publication of which have been rendered possible by the generosity of a number of American contributors to a specially-raised fund. Sir T. Heath, who writes the introduction, agrees that "there could not be a more appropriate time for the appearance of a series such as that for which the present volume forms a part." Those who are acquainted with Prof. Smith's *Teaching of Elementary Mathematics* and of his work (with Mr. Karpinski) on *Hindoo Arabic Notation*, will be aware of his devotion to the historical side of mathematics, and his views as to the advantages that an acquaintance with history may be to the judicious mathematical teacher. In the present little volume of 164 pages we find the same wealth of interesting detail. He points out many survivals of epithets or phrases, now merely of technical use, which once had a vital meaning. In estimating the influence of the Roman abacus, he concludes that the later form of it found its way in some unknown manner to China, and hence to Japan, where in the form of the *soroban* it is still in use, and "likely to remain so until that country, like the West, adopts some convenient and inexpensive form of the more elaborate computing machine."

The work can be strongly recommended, both to teachers interested in the history of their profession and to those outside it who from time to time urge the adoption of some improved method of their own discovery. The latter class may find from Prof. Smith's references that the most hopeful of these discoveries are usually to be found in the writings of Plato.

EDWARD M. LANGLEY.

Cours de Mathématiques Générales, à l'usage des étudiants en Sciences naturelles. By G. VERRIEST. Part I.: **Calcul Différentiel. Géométrie Analytique à Deux Dimensions.** Pp. 337. 38 fr. 1923. (Éditions Universitas, Louvain; Gauthier-Villars, Paris.)

The first nine chapters of this book form the most careful, thorough, and

illuminating exposition, on elementary lines, of the principles of the differential calculus that we have seen. The account of limits, derivatives, and differentials, and their use, is one which could be recommended as a first course to the future pure mathematician as well as to the scientist. A very good feature of the work is the way in which the student is carefully and repeatedly shown that the differential, though not always the actual variation, is the principal part of that variation, or, in other words, the part which enters *effectively* into the calculation of limits of ratios and of sums. A correct presentation of $ds^2 = dx^2 + dy^2$ in a book for science students is indeed rare: the author achieves this without, we think, any great strain on the average reader's attention. Geometrical intuition is frequently used, but the reader is expressly told that the method, though free from error in the elementary discussions here entered upon, is not a reliable one for conducting a searching examination into the properties of any and every function.

The chapter on partial derivatives contains one or two lapses from the high standard of clearness and accuracy which mark the book: curiously enough, the succeeding chapter on total differentials is extremely well done and avoids the errors of the previous similar work. The principles of analytical geometry and a number of problems on the straight line and well-known loci are given early in the book; the systematic tracing of curves given by $y = f(x)$ and the essentials of an analytic treatment of conics come later: all are very clearly dealt with.

Throughout the book, the applications of theory, both to mathematical and physical problems, are excellent. The varied physical examples we found most interesting—the use of a diaphragm in a camera as a means of obtaining a "sharp" picture is a pleasant variant to some of the older questions on approximative work with Maclaurin's formula. The author has shown himself to be an excellent teacher; he has realised that a text-book of this sort must not only map out the path, but also help the reader along it, must not merely set out the facts but aid the student to understand them.

On one or two points of detail it is easy enough to criticise, but, taken as a whole, the book deserves the attention of all serious science students who feel that their knowledge of the calculus is not what it should be.

Les Fonctions Circulaires et les Fonctions Hyperboliques, étudiées parallèlement en partant de la définition géométrique. By H. Tripier. Pp. iv + 56. 5 fr. 1923. (Vuibert.)

The first six chapters contain an interesting parallel geometric treatment of the functions for real arguments, the definition of the hyperbolic functions being that indicated in *Chrystal* (1906), ii. p. 311. The equivalent series are obtained from Maclaurin's formula (quoted as known), and these series are used to define the functions for complex arguments and to express them in exponential forms.

The book is addressed primarily to engineers, and the geometric definition of the hyperbolic functions would probably be helpful to this class of student, to whom the more usual definitions often appear unreal. On the other hand, the exposition is not all it might be; some parts are rendered difficult by too brief statement, while the inverse functions are nowhere dealt with clearly. That the inverse functions are many-valued is recognised and used several times in the course of the text: the periodicity of the functions, for all arguments, appears in a short concluding chapter. Chapter VII. contains an ingenious graphical method of finding a value of such functions as

$$\arg \sinh (x + yi).$$

W. L. Farrer.

From Determinant to Tensor. By W. F. Sheppard. Pp. 127. 8s. 6d. 1923. (Clarendon Press.)

This is quite a timely little book, meant to initiate the beginner into the notation and theory of Tensors. The author, being of opinion that determinants afford the most appropriate entrance-hall for getting at the newer subject, opens with an elementary sketch of the theory of them, indeed devotes almost quite the half of his page-space to them. Five chapters bear the heading "Determinants," and the other five the heading "Sets"; but in the

last of the former five there is a gradual edging-in of the latter subject such as the professional teacher approves of.

The section on Determinants is of the auxiliary kind usual in books on the Theory of Equations, and is intermediate in extent between Todhunter's and Burnside and Panton's. In construction and style it has, perhaps, less of text-book formality, and differs from both in having no exercises. One or two fresh usages of words jar a little, for example, "minor" spoken of only as "minor of an element," and "transposed" used to displace something else and used as a noun.

In the other section the first chapter naturally deals with "Sets of Quantities," the second with "Related Sets of Variables," the third with "Differential Relations of Sets," the fourth—a very interesting chapter—with "Examples from the Theory of Statistics," and the fifth with "Tensors in the Theory of Relativity."

The section on Determinants, we are inclined to think, might with no loss have been condensed a little, and the space so saved given with advantage to the other section, where additional examples of applications would have been helpful. Even a few well-chosen exercises would not have been amiss, in order to familiarise the reader with the thick growth of fresh symbolism. These, and other allied matters, however, will doubtless be settled in the course of a tutor's actual use of the book with different classes of readers; and the second edition can then show the fruits of the experience.

In any case, the preparation of such a work was a "happy thought," and it deserves a cordial welcome. T. M.

Shop Mathematics. By J. M. Christman. Pp. x + 316 + lxviii. 10s. net. 1923. (Macmillan.)

This book commences at an early stage of mathematical training, and progresses sufficiently slowly to enable the unaided pupil to find his way.

"Short-cuts" are lucidly treated; "Shop Practice," though limited in quantity, is excellent, especially the treatment of milling and hobbing. The greater portion of the book is devoted to gearing, and the student who works through the examples given should have no difficulty with any gearing problems.

The printing, diagrams and tables are excellent, but the price is too high to command a ready sale among those for whom it was intended.

Mechanical Engineering Formulae. By E. W. Huddy. Pp. 176. 4s. 6d. net. 1923. (E. and F. N. Spon.)

The work is concise, well printed and may be confidently recommended to the university student or technical engineer.

The Saturated Steam Table, Mean Specific Heat of Superheated Steam, Properties of Dry Saturated Steam are all valueless to the engineer, in that the source of these nine pages of data is not given.

More attention might have been given to the indexing, as the value of a book of this nature depends very largely on the rate at which information may be obtained from it. F. C. M.

Prolegomena to Analytical Geometry in Anisotropic Euclidean Space of Three Dimensions. By E. H. Neville. Pp. xxii + 367. 30s. net. 1922. (Cambridge University Press.)

I could wish that the task of reviewing this important book had fallen into more competent hands; but I have had no difficulty in seeing that it is extraordinarily good, and feel that if my qualifications as a critic were higher its merits would appear to me still more conspicuous. It possesses, I think, all the qualities needed to give distinction to a work in this *genre*. An immense amount of laborious thinking must have gone to its making, yet it is never heavy or dull; it is based upon wide learning, but the learning is felt in the fibre, not exhibited for admiration; the diction is economical, lucid and precise; and the mathematical craftsmanship is of the highest order. Above all, the exactness in detail which is the special characteristic and in truth the special aim of the work expresses not the niceness of a pedant who worries too much *de minimis*, but the intellectual passion of a man who, like

the ideal metaphysician, is obstinately determined to think clearly about everything he handles. Thus the work has a warmth and movement which must make its reading a delightful tonic to any one who has a sense for the aesthetic value as well as the ultimate practical importance of clear and precise notions about mathematical entities and their relations. It is not irrelevant to add that the text is well arranged and beautifully printed, and that the student's progress is facilitated by a most careful system of marginal numeration of the sections and smaller elements of the argument, and by full references from one part of the argument to another.

The work, as a whole, falls into two main parts. Throughout the first part (Bks. I., II., III.) we are in the ordinary Euclidean space of the text-books on analytical geometry, and the aim is to place on a firm foundation the theory of vector analysis and the use of Cartesian frames of reference. In the second part (Bks. V., VI.) we pass into the "algebraic" and "ideal" spaces constructed conceptually by mathematicians upon the model (itself, of course, conceptual) of ordinary Euclidean space.

Bk. I. deals with familiar and very elementary matters—directions and angles, measurement by steps, and parallel projection. But although the topics are simple, the precision and subtlety of the treatment prepare the reader's mind usefully for what is to be the prevalent temper of the work. Gauss's unit sphere is introduced as a means of representing directions, and careful consideration is given to the conventions in virtue of which rays and angles, the areas of triangles and the volumes of tetrahedra may acquire signs. The terms "prepared line," "prepared plane," "prepared space" are used to distinguish lines, planes and spaces qualified by these conventions from those unqualified: for instance, a triangle has a sign only when it lies in a prepared plane. The definition of a step as a pair of ordered points (not of some entity supposed to be determined by the points) is a simple illustration of the care with which the author invariably relates complex concepts to perfectly definite fundaments. Certain characters assigned to steps have importance mainly in reference to the subsequent theory of vectors: namely, (i) that a proper step (*i.e.* a step which is not zero) has, in spite of the fact that it is an *ordered* point-pair, a forward and a backward direction, and (ii) that a zero step has all directions. Readers who remember the discussion of zero in Russell's *Principles* may boggle at the second of these conventions and protest that there is an actual if subtle difference between the concepts (say) of a zero step to the north and a zero step to the east. In Bk. II., however, Prof. Neville gives what I feel bound to recognise as an adequate defence of the convention he adopts, and it is still more fully justified in its subsequent use.

The second book, which contains the theory of vectors and rotors, is a most interesting and satisfying piece of work, and I think that Prof. Neville must have enjoyed peculiar pleasure in writing it. He begins with a novelty at which I have already glanced. According to the Hamiltonian definition, a vector consists of a signless real number associated with a single direction, but mathematicians have often found it convenient to depart from the classical conception and to make use of the idea of a vector with a negative amount. In order to legitimise such practices Prof. Neville defines a vector at the outset as consisting of two real numbers, one the opposite of the other, associated respectively with a direction and its reverse. This is the concept which he carries through his excellently lucid and illuminating discussion of the nature and fundamental properties of vectors, rotors and couples. From Poinsot's well-known theorem he derives the concept of a "Poinsot set" consisting of a rotor and a couple whose momental vector is parallel to the rotor, and from the equivalence of such sets the more abstract concept of a rotor associated with a parallel momental vector—to which, following Clifford, he gives the name "motor." In his elaboration of these subjects Prof. Neville has, I think, clarified, systematised and extended the work of his predecessors in a very notable manner. It is, perhaps, unnecessary to add that he has treated them in the spirit of a pure geometer, leaving on one side their baser uses in applied mathematics.

Book II. ends with a valuable critical note on the uses of the word "sum" in vector analysis. In the case of a set of vectors or of motors there is always

a definite vector or motor which may naturally be defined as their sum. But this is not true of sets of rotors or of couples; the concept of the sum of such a set must, therefore, be formed in a less simple and direct way. The method chosen by Prof. Neville is the one by which Frege and Russell independently solved the difficult problem of defining a number. Aggregates of things are said to contain the same number of items when the items in any two aggregates can be brought into one to one correspondence with one another; but, unfortunately, this simple explanation supplies no answer to an enquirer who wants to know what a number actually is. It is useless to say that it is the common property of all the similar aggregates, for a dialectician may show that they have several common properties. The Frege-Russell method of avoiding the difficulty is to say that the common number of the aggregates is the class composed of the whole of them. Similarly, if a finite set of couples be given, there is a definite class composed of all the couples that are equivalent to it, and this class is defined by Prof. Neville to be their sum. I have deemed it worth while to explain the method, since it is used elsewhere in the work, and the reader is assumed to be acquainted with it.

Book III. begins with a chapter on the trigonometry of plane and spherical triangles with directed sides, in which certain valuable novelties in technical treatment are introduced. The results are put to use in the following chapters on Cartesian axes and vector frames. Prof. Neville's handling of these subjects is remarkably fresh, vigorous and philosophic, and, I think, full of originality. Its spirit is, perhaps, best conveyed briefly by his statement (p. 122) that projected products (*i.e.* the scalar products of Hamilton with sign reversed) "are the very stuff of which analytical geometry is made." It should be added that Prof. Neville's technical methods enable him to achieve with relative ease a generality which frees vector analysis from the traditional dominance of the trirectangular frame and makes the Hamiltonian $\mathbf{i}, \mathbf{j}, \mathbf{k}$ superfluous. With regard to the architectural ideas underlying this part of the work, it must suffice to say that the doctrine of vector frames bears to the doctrine of Cartesian frames much the same relation as vector-theory bears to the theory of steps. The former doctrine is based upon the directions associated with definite lines; the latter, more abstract in character, is based upon directions only. This is one reason why vector frames are of special value in differential geometry.

In ordinary analytical geometry space is taken for granted as something already existing and enjoying the properties specified in the axioms, and our aim is to study its further properties by the method of associating numbers with the spatial elements. But it is possible to drop the original space out of sight, and to conceive an aggregate consisting of numbers only in which, nevertheless, the properties of actual or geometrical space are reproduced. Such an aggregate is an "algebraic space." The numbers involved in the concept may be real or complex, and the resulting "space" will accordingly be real or complex algebraic space. It is obvious that the use of geometrical terms in discussing these artefacts is pure metaphor; nevertheless, the discussion may achieve results of value for ordinary geometry. For instance, the study of real algebraic space establishes the consistency of the Euclidean system, while owing to the fact that, in Prof. Neville's phrase, analysis can work most freely in the field of complex numbers, the study of complex algebraic space may reveal features capable of transference through real algebraic space to ordinary geometry.

The construction of algebraic space is the subject of Book IV. The assumption of position and direction as undefinables, which is the natural starting-point of ordinary geometry, would be here out of place; the author takes, therefore, the ideas of a vector and a point as undefined, and develops his theme from six assumptions which determine the relations between vectors and vectors and between vectors and points. An essential part in the development is played by the ideas of a "vecline," a "vecplane" and a "vecspace," which Prof. Neville first defined in his illuminating tract, *The Fourth Dimension*.* The Book ends with a brief discussion of the part played by complex geometry in the study of real space.

* Cambridge University Press, 1921.

In the last Book the device of extending the natural language of ordinary geometry to the analogous properties of artificial constructions receives a further typical extension, due in essence to von Staudt. Since a sheaf of ordinary lines passing through a point may be regarded as determining that point, we may treat the sheaf itself as an element in a new geometry and speak of it as an "ideal point." A set of parallel lines may then also be regarded as constituting an ideal point and be distinguished from the former ideal point by calling it a point "at infinity." The historical development of geometry makes it difficult to avoid terms, such as this, about which a good deal of primitive fog still lingers in the minds of students and their teachers, but Prof. Neville takes care to remind his readers that there is no pretence of the actual existence of ordinary points that have somehow become inaccessible. The concept of ideal points having thus been formed in the field of actual geometry it may be carried over into algebraic space, where it serves as the basis of a new analytic development. A very important part of that development, as it is given in these pages, deals with the theory of circles and their intersections. There is, I think, no part of the work where the power of Prof. Neville's methods and the beautiful economy of his treatment are more strikingly exhibited.

I have been obliged to content myself with indicating baldly the contents and the general spirit of a volume which the author offers modestly as a preface to the *Principes* of M. Darboux. I hope, however, that I have made sufficiently manifest my opinion that it is a masterly treatise, full of instruction and inspiration for a serious student, abounding in technical excellencies and advances which an expert may study with interest and profit, and altogether a notable contribution to British mathematical scholarship.

T. P. NUNN.

Cours de Mécanique Céleste. Vol. I. By M. H. ANDOYER. Pp. vi + 440. 50 fr. 1923. (Gauthier-Villars.)

For twenty years it has fallen to the lot of Prof. Andoyer to give an annual course, each lasting some six months and covering the ground that must be traversed by all who hope to embark with success upon the work of astronomical computations—a field in which the author of the book before us has won for himself the highest reputation. His aim is to give as simply and fully as possible to computers and practical astronomers the practical solutions afforded by Astronomy to the real problems of Celestial Mechanics. As he tells us in his Preface, to facilitate his task he confines himself almost exclusively to a development in all necessary detail of the methods which lead to the simplest and most trustworthy calculations. In the new and revised edition of his great *Cours d'Astronomie*, he has made a special point of utilising problems borrowed from reality to illustrate the numerical application of the formulae he establishes—*e.g.* the lunar eclipse of Feb. 8, 1925, the transit of Mercury of May 7, 1924, the solar eclipse of January 24, 1925. This excellent practice he has followed in the present treatise, and the resulting impression made upon the student by this contact with actualities cannot fail to be effective and lasting. The volume is divided into three books dealing respectively with the general problems of celestial mechanics ; the practical study of the Keplerian orbit ; planetary theory. Among the fundamental problems to which Prof. Andoyer has restricted his attention are those arising out of the theory of the determination of the orbits of minor planets and comets. These, as he says, are often dealt with in works on Terrestrial Astronomy, but they certainly belong to the domain of Celestial Mechanics, and for that and other reasons he deals with them fully in these pages. A section is devoted to the method of least squares, and some 60 pages are given to interpolation theory before the student attempts the calculation of perturbations by numerical quadratures. Vol. II. is to deal with the theory of the moon, the rotations of the earth and moon, and the theory of the Galilean satellites of Jupiter. The whole promises to be a masterly contribution to the literature of the subject.

THE LIBRARY,

160 Castle Hill, Reading.

ADDITIONS.

The Librarian reports the following gifts:

From Mr. W. J. Greenstreet (third list):

G. B. Airy	Six Lectures on Astronomy. 2nd ed.	c.1850
	Gravitation	1834
H. Andoyer	Cours de Mécanique Céleste. Tome I.	1923
H. F. Baker	Principles of Geometry. Vol. I. *Have other members later volumes to give?*	1922
Lord Brougham	Lives of Men of Letters and Science	1845
L. Couturat	Algebra of Logic. Trans. (from French into English) by L. G. Robinson. Preface by P. E. B. Jourdain	1914
A. Harnack	Calculus. Trans. (from German into English) by G. L. Cathcart	1891
W. Hooper	Rational Recreations (4 vols.)	1794
C. Huygens	Œuvres. Tomes 4, 9, 10, 11, 14 *Odd volumes of the definitive edition that is being issued by the Société Hollandaise des Sciences. Other odd volumes would be welcomed.*	1891-1920
V. E. Johnson	Uses and Triumphs of Mathematics	[1889]
P. S. de Laplace	Théorie Analytique des Probabilités *This copy has since been bound for the Library in contemporary calf.*	1812
H. Laurent	Traité d'Analyse. Vol. 2: Calcul Différentiel; Applications Géométriques *Of this monumental treatise, the Library still lacks* Vols. 1, 5, 6, 7.	1887
A. Ram	Problems in Dynamics	(1916)
A. A. Robb	Theory of Time and Space	1914
J. S. Russell	Geometry in Modern Life *The substance of two lectures to the Eton Literary Society.*	1878
E. Stone	Analise des Infiniment Petits. Trans. (from English into French) by — Rondet	1735
W. Trail	Life of Robert Simson	1812
H. Vogt	Mathématiques Supérieures	1901
J. Ward	Young Mathematician's Guide	1771
E. Waring	Proprietates Algebraicarum Curvarum	1772
	Meditationes Algebraicæ	1782
	Meditationes Analyticæ	1785
J. Wolfson	Visual Arithmetic	1896
J. R. Young	Algebraical Equations	1843

From Mr. T. W. Hope:

C. Burch	Short Cuts and Bye-Ways in Arithmetic	1903

Also school text-books by:

Hobson and Jessop, Lane and Lane, Layng, Loney, Tuckey.

From Rev. J. J. Milne:

	The Penny Cyclopædia. 30 vols.	1833-1858
W. Jones	Conic Sections	1774
J. J. Milne	Cross-Ratio Geometry	1911

REQUESTS.

Anyone able and willing to lend one of the following books is asked to communicate with the Librarian:

Klein	Abhandlungen über den Mathematischen Unterricht in Deutschland.
Klein und Schummach	Der Mathematischen Unterricht an den Höheren Schulen.

NOTICE.

Mr. E. M. Langley will be glad if any reader of the *Mathematical Gazette* who is using, or has used, stereoscopic slides designed by him for illustrating Solid Geometry, will communicate with him at 48 Waterloo Road, Bedford.

YORKSHIRE BRANCH.

A meeting of the Branch was held on Saturday, 24th November, at University House, Leeds, Mr. W. F. Beard of Wakefield Grammar School presiding. The meeting expressed its very warm appreciation of the work of the Rev. A. V. Billen, the Secretary of the Branch, who was retiring under the "three years' rule" of the Branch. Mr. S. Lister of West Leeds High School was elected to succeed him. The Rev. A. V. Billen of Leeds Grammar School, Mr. S. H. Stelfox, H.M.I., Mr. J. H. Blacklock of Rotherham Grammar School, and Mr. R. W. Evans of Ilkley Grammar School were elected to replace the retiring members of the Executive Committee.

A very interesting address on "Mathematics in Education" was then given by Mr. W. P. Welpton of Leeds University. By tracing the growth of the subject as a tool or instrument for dealing with "things" collectively and individually, with space and time and with force and matter, he showed how the creative genius of mankind had evolved a body of ideas and principles which were the intellectual tools necessary for thinking accurately and acting effectively in the realms of things and of human activities. The practical or empirical side always preceded the abstract or theoretical side both as to content and language, and the necessity of following the same course in the teaching of the subject was exemplified by excellent illustrations and models. The aesthetic side came with power in using the tool and in the polishing and perfecting of it. An illuminating discussion followed.

Dr. S. Brodetsky afterwards related some interesting experiences connected with mathematical matters which he had gained during his recent journeyings in Central Europe.

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217. The mathematical inquirer must learn to substitute for his own private and momentary use, abbreviations which could not be tolerated in the final expression of results.—De Morgan [Symbol] *P.C.*

De Morgan adds to each side of an equation, *i.e.* does not "change side, change sign."

218. A man cannot know modes of life as well in Minorca as in London; but he may study mathematics as well in Minorca.—Boswell, April 7, 1778.

219. When Johnson felt his fancy, or fancied he felt it, disordered, his constant recurrence was to the study of arithmetic.—Piozzi, *Anec.*, *Johnsonian Misc.*, Birkbeck Hill, i. p. 200.

220. The Asses' Bridge—Epigram, 1780:

"If this be rightly called the bridge of asses,
It's not the fool that sticks, but he that passes."

Aliter. "Although the asses' bridge was made for asses,
It's not the ass that sticks, but he that passes."

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AT THE UNIVERSITY PRESS, GLASGOW.

The Mathematical Association.

President:—

Sir Thomas L. Heath, K.C.B., K.C.V.O., D.Sc., F.R.S.

The Annual Meeting of the Mathematical Association will be held at the London Day Training College, Southampton Row, London, W.C. 1, on Monday, 7th January, 1924, at 5.30 p.m., and Tuesday, 8th January, 1924, at 10 a.m. and 2.30 p.m.

MONDAY EVENING, 5.30 p.m.

1. "Earthquakes," by Professor H. H. Turner, D.Sc., F.R.S.

TUESDAY MORNING, 10 a.m.

BUSINESS:

2. The Report of the Council for the year 1923.
3. The Treasurer's Report for the year 1923.
4. The Reports of the Teaching Committees.
5. The Election of Officers and Members of the Council for the year 1924.

PAPERS AND DISCUSSIONS:

6. Mr. W. C. Fletcher, on "Mathematics and English."
7. Mr. W. Hope-Jones, on "A Plea for Teaching Probability in Schools."
8. Mr. A. W. Lucy will exhibit a Surveying Instrument and explain its use in connection with practical work in Trigonometry.

TUESDAY AFTERNOON, 2.30 p.m.

9. Sir Thomas L. Heath, K.C.B.: The President's Address.
10. Professor E. H. Neville will open a Discussion on the Report on the Teaching of Geometry.
11. Mr. G. Goodwill, on "Euclid and his Successors: some confusion and a way out."
12. Professor C. Godfrey, M.V.O., on "Construction in Geometry. What is legitimate?"

C. PENDLEBURY, } *Honorary*
MARGARET PUNNETT, } *Secretaries.*

THE MATHEMATICAL ASSOCIATION.

(*An Association of Teachers and Students of Elementary Mathematics.*)

"*I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament thereunto.*"—BACON.

THE MATHEMATICAL ASSOCIATION, which was founded in 1871, as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

Its purpose is to form a strong combination of all persons who are interested in promoting good methods of teaching mathematics. The Association has already been largely successful in this direction. It has become a recognised authority in its own department, and has exerted an important influence on methods of examination.

The Annual Meeting of the Association is held in January. Other Meetings are held when desired. At these Meetings papers on elementary mathematics are read and discussed.

Branches of the Association have been formed in London, Southampton, Bangor, Yorkshire, Bristol, Manchester, Cardiff, Sydney (New South Wales), and Queensland (Brisbane). Further information concerning these branches can be obtained from the Honorary Secretaries of the Association.

"**The Mathematical Gazette**" (published by Messrs. G. BELL & SONS, LTD.) is the organ of the Association. It is issued at least six times a year. The price per copy (to non-members) is usually 2s. 6d. each. The *Gazette* contains—

(1) ARTICLES, mainly on subjects within the scope of elementary mathematics;

(2) NOTES, generally with reference to shorter and more elegant methods than those in current text-books;

(3) REVIEWS, written by men of eminence in the subject of which they treat. They deal with the more important English and Foreign publications, and their aim, where possible, is to dwell on the general development of the subject, as well as upon the part played therein by the book under notice;

(4) SHORT NOTICES of books not specially dealt with in the REVIEWS;

(5) QUERIES AND ANSWERS, on mathematical topics of a general character.